On multiplicatively badly approximable numbers

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Abstract

The Littlewood Conjecture states that $\liminf_{q\to\infty} q\cdot||q\alpha||\cdot||q\beta|| = 0$ for all $(\alpha,\beta)\in\mathbb{R}^2$. We show that with the additional factor of $\log q\cdot\log\log q$ the statement is false. Indeed, our main result implies that the set of (α,β) for which $\liminf_{q\to\infty} q\cdot\log q\cdot\log\log q\cdot||q\alpha||\cdot||q\beta|| > 0$ is of full dimension.

1 Introduction

The famous Littlewood conjecture (LC) states that for any pair of real numbers (α,β)

$$\liminf_{q\to\infty} q\cdot||q\alpha||\cdot||q\beta|| = 0 \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. Equivalently, the set

$$\{(\alpha,\beta)\in\mathbb{R}^2\ :\ \liminf_{q\to\infty} q\cdot||q\alpha||\cdot||q\beta|| > 0\} \tag{2}$$

is empty. This problem was conjectured in 1930's and it is still open. For recent progress concerning this fundamental problem see [4, 6] and references therein. It is easily seen that (1) holds for all $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}$ outside the set **Bad** of badly approximable numbers defined as follows

$$\mathbf{Bad} := \{\alpha\in\mathbb{R}\ :\ \liminf_{q\to\infty} q||q\alpha|| > 0\}.$$

In attempt to understand what should be a proper analogue of badly approximable points in multiplicative case several authors investigated the following set (we will follow the notation introduced in [2]). For $\lambda\geqslant 0$ let

$$\mathbf{Mad}^\lambda := \{(\alpha,\beta)\in\mathbb{R}^2\ :\ \liminf_{q\to\infty} (\log q)^\lambda\cdot q\cdot||q\alpha||\cdot||q\beta|| > 0\}.$$

In other words, $\mathbf{Mad}^\lambda$ is a modification of the set in (2) such that the corresponding condition is weakened by $(\log q)^\lambda$. More generally, given a function $f\ :\ \mathbb{N}\to\mathbb{R}^+$, define the set

$$\mathbf{Mad}(f) := \inf\{(\alpha,\beta)\in\mathbb{R}^2\ :\ \liminf_{q\to\infty} f(q)\cdot q\cdot||q\alpha||\cdot||q\beta|| > 0\}. \tag{3}$$

In [2] the author and Velani conjectured that

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Conjecture A (BV)

$$\mathbf{Mad}^\lambda = \emptyset \quad \textit{for any } \lambda < 1,$$
$$\dim(\mathbf{Mad}^\lambda) = 2 \quad \textit{for any } \lambda \geqslant 1$$

where $\dim(\cdot)$ denotes the Hausdorff dimension. If true this conjecture implies that the proper multiplicative analogue of the set **Bad** is $\mathbf{Mad}^1$. Note that LC is equivalent to the statement that $\mathbf{Mad}^0$ is empty. Therefore BV conjecture implies LC. Regarding the first part of BV conjecture all that is known to date is the remarkable result of Einsiedler, Katok and Lindenstrauss [4] which states that $\dim \mathbf{Mad}^0 = 0$. On the other hand according to the second part the best known result is due to Bugeaud and Moschevitin [3]. It states that $\dim \mathbf{Mad}^2 = 2$. So we have a gap $0 \leqslant \lambda < 2$ where the behavior of $\mathbf{Mad}^\lambda$ is completely unknown.

In this paper we will address the second part of the BV conjecture. In particular, we will show that

$$\dim \mathbf{Mad}(f) = 2 \quad \text{if} \quad f(q) = \log q \cdot \log\log q.$$

It will straightforwardly imply that $\dim(\mathbf{Mad}^\lambda) = 2$ for any $\lambda > 1$.

It is worth mentioning that the 'mixed' analogue of this result was achieved recently by author and Velani. It was proven that the set

$$\mathbf{Mad}_{\mathcal{D}}(f) := \{\alpha \in \mathbb{R} \ : \ \liminf_{q\to\infty} f(q)\cdot q \cdot |q|_{\mathcal{D}}||q\alpha|| > 0\}$$

has full Hausdorff dimension. All the details can be found in [2].

1.1 Simultaneous and dual variants of Mad

It is well known that Littlewood conjecture has an equivalent formulation in terms of linear forms. In other words, (1) is equivalent to the statement that

$$\liminf_{|AB|\to\infty} |A|^*|B|^* \cdot ||A\alpha - B\beta|| > 0$$

where $|x|^* := \max\{|x|, 1\}$. However it is not known if (3) can be reformulated in the same manner. In other words, define the sets

$$\mathbf{Mad}_L(f) := \inf\{(\alpha,\beta) \in \mathbb{R}^2 \ : \ \liminf_{|AB|\to\infty} f(|A|^*|B|^*)\cdot |A|^*|B|^*||A\alpha - B\beta|| > 0\} \qquad (4)$$

and

$$\mathbf{Mad}_L^\lambda := \mathbf{Mad}_L(\log^\lambda q).$$

Then $\mathbf{Mad}(f)$ and $\mathbf{Mad}_L(f)$ are not necessarily the same. However as it will be shown in the next sections these sets are closely related to each other. For consistency in further discussion we will use the notation $\mathbf{Mad}_P^\lambda$ and $\mathbf{Mad}_P(f)$ instead of $\mathbf{Mad}^\lambda$ and $\mathbf{Mad}(f)$ respectively. It will reflect the fact that in one case we deal with points and in another case we deal with lines.

It appears that instead of investigating $\mathbf{Mad}_P(f)$ and $\mathbf{Mad}_L(f)$ independently it is easier to deal with them simultaneously. In particular, we prove the following result:

Theorem 1 *Let $f(q) = \log q \cdot \log\log q$. Then*

$$\dim(\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f)) = 2.$$

1.2 Main result

For convenience, we define the 'modified logarithm' function $\log^* : \mathbb{R} \to \mathbb{R}$ as follows

$$\log^* x := \begin{cases} 1 & \text{for } x < e; \\ \log x & \text{for } x \geqslant e. \end{cases}$$

From now on

$$f(q) := \log^* q \cdot \log^* \log q.$$

The key to establishing Theorem 1 is to investigate the intersection of the sets $\mathbf{Mad}_P(f)$ and $\mathbf{Mad}_L(f)$ along fixed vertical lines in the (x, y)-plane. With this in mind, let L_x denote the line parallel to the y-axis passing through the point $(x, 0)$.

The following constitutes our main theorem.

Theorem 2 *For any* $\theta \in \mathbf{Bad}$

$$\dim(\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \cap \mathrm{L}_\theta) = 1 \ .$$

Since by Jarník (1928) the Hausdorff dimension of **Bad** is one, Theorem 1 can be easily derived from Theorem 2 with the help of the following general result that relates the dimension of a set to the dimensions of parallel sections, enables us to establish the complementary lower bound estimate – see [5, pg. 99].

Proposition *Let F be a subset of $\mathbb{R}^2$ and let E be a subset of the x-axis. If* $\dim(F \cap \mathrm{L}_x) \geqslant t$ *for all $x \in E$, then* $\dim F \geqslant t + \dim E$.

Indeed, let $F = \mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f)$ and $E = \mathbf{Bad}$. In view of $\dim(\mathbf{Bad}) = 1$ and Theorem 2, one gets $\dim \mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \geqslant 2$. Since $\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \subset \mathbb{R}^2$, the upper bound statement for the dimension is trivial. Therefore the main ingredient in establishing Theorem 1 is Theorem 2.

Regarding the proof of Theorem 2 we will use ideas similar to those in [2] which firstly appeared in joint work of author, Pollington and Velani [1]. However the technical details in this paper are substantially more complicated than those in [2].

2 Preliminaries

Let S be any subset of $\mathbb{R}^2$. By S_θ we denote its orthogonal projection onto the line L_θ. Let $P(p, r, q) := (p/q, r/q)$ be a rational point where $(p, r, q) \in \mathbb{Z}^3, \gcd(p, r, q) = 1$. Denote by the height of P the value

$$H(P) := q^2|q\theta - p| \geqslant q^2||q\theta||.$$

Denote by $\Delta(P, \delta)$ the following segment on L_θ:

$$\Delta(P, \delta) := \{\theta\} \times \left(\frac{r}{q} - \frac{\delta}{H(P)}, \frac{r}{q} + \frac{\delta}{H(P)} \right).$$

So $|\Delta(P, \delta)| = 2\delta H(P)^{-1}$.

Given a line with integer coeffitients

$$L(A, B, C) := \{(x, y) \in \mathbb{R}^2 \ : \ Ax - By + C = 0\},$$

$$(A, B, C) \in \mathbb{Z}^3, \ B \neq 0, \ \gcd(A, B, C) = 1 \tag{5}$$

denote by the height of L the value

$$H(L) := |A|^* B^2.$$

Denote by $\Delta(L,\delta)$ the following segment on L_θ:

$$\Delta(L,\delta) := \{\theta\} \times \left(\frac{A\theta + C}{B} - \frac{\delta}{H(L)}, \frac{A\theta + C}{B} + \frac{\delta}{H(L)}\right).$$

So $|\Delta(L,\delta)| = 2\delta H(L)^{-1}$.

Given constants $c > 0$ and $Q > 0$ define the auxiliary sets:

$$\mathbf{Mad}_P(f,c,Q) := \left\{(\alpha,\beta) \in \mathbb{R}^2 \ : \ f(q)\cdot q \cdot ||q\alpha|| \cdot ||q\beta|| > c \ \ \forall q \in \mathbb{N}, \ \geqslant Q\right\}$$

and

$$\mathbf{Mad}_L(f,c,Q) := \inf\left\{(\alpha,\beta) \in \mathbb{R}^2 \ : \ \begin{array}{l} f(|A|^*|B|^*)\cdot |A|^*|B|^*||A\alpha - B\beta|| > c, \\ \forall (A,B) \in \mathbb{Z}^2, \ |A|^*B^2 \geqslant Q \end{array}\right\}.$$

It is easily verified that $\mathbf{Mad}_P(f,c,Q) \subset \mathbf{Mad}_P(f)$, $\mathbf{Mad}_L(f,c,Q) \subset \mathbf{Mad}_L(f)$ and

$$\mathbf{Mad}_P(f) \cap \mathbf{Mad}_L(f) \ = \bigcup_{c>0} (\mathbf{Mad}_P(f,c,Q) \cap \mathbf{Mad}_L(f,c,Q)) \ .$$

For convenience we will omit the parameter Q where it is irrelevant and write $\mathbf{Mad}_P(f,c)$ and $\mathbf{Mad}_L(f,c)$ for $\mathbf{Mad}_P(f,c,Q)$ and $\mathbf{Mad}_L(f,c,Q)$ respectively.

So it suffices to prove that the set $\mathbf{Mad}_P(f,c) \cap \mathbf{Mad}_L(f,c) \cap \mathrm{L}_\theta$ has full Hausdorff dimension for some positive constant c.

Geometrically, the set $\mathbf{Mad}_P(f,c)$ consists of points that avoid the "neighborhood" of each rational point $P = (p/q, r/q)$ defined by the inequality

$$\left|x - \frac{p}{q}\right|\left|y - \frac{r}{q}\right| < \frac{c}{f(q)q^3}.$$

This "neighborhood" of P will remove the interval $\Delta(P, cf(q)^{-1})$ from L_θ. Without loss of generality we can assume that $|q\theta - p| = ||q\theta||$. Otherwise we just replace the point P by $P' := (p'/q, r/q)$ such that $|q\theta - p'| = ||q\theta||$. Then $\Delta(P') \supset \Delta(P)$ and the "neighborhood" of P will not remove anything more than one of P'.

Similarly one can show that the set $\mathbf{Mad}_L(f,c)$ consists of points that avoid the "neighborhood" of each line $L(A,B,C)$ defined by

$$|Ax - By + C| < \frac{c}{f(|A|^*|B|^*)|A|^*|B|^*}$$

where the coefficients A, B, C satisfy $(A,B) <> (0,0)$ and $\gcd(A,B,C) = 1$. For $B = 0$ it leads to the following inequality:

$$||Ax|| < \frac{c}{|A|f(|A|)}.$$

Take $c < \inf\limits_{q\in\mathbb{N}} q||q\theta||$. Then this inequality is not true for $x = \theta$, in other words the "neighborhood" of the line do not remove anything from L_θ. Therefore it is sufficient to consider the lines $L(A,B,C)$ with $B \neq 0$, so the coefficients (A,B,C) will satisfy (5). Then the "neighborhood" of $L(A,B,C)$ will remove the interval $\Delta(L, cf(|A|^*|B|^*)^{-1})$ from L_θ.

2.1 Cantor sets

In the proof we will use the general Cantor framework firstly introduced in [2]. Here we reproduce the definitions and facts which will be used in later discussion. For more details we refer to the paper [2].

Let I be a closed interval in $\mathbb{R}$. Let $\mathbf{R} := (R_n)$ with $n \in \mathbb{Z}_{\geqslant 0}$ be a sequence of natural numbers and $\mathbf{r} := (r_{m,n})$ with $m, n \in \mathbb{Z}_{\geqslant 0}$, $m \leqslant n$ be a two parameter sequence of non-negative real numbers.

The construction. We start by subdividing the interval I into R_0 closed intervals I_1 of equal length and denote by $\mathcal{I}_1$ the collection of such intervals. Thus,

$$\#\mathcal{I}_1 = R_0 \qquad \text{and} \qquad |I_1| = R_0^{-1}\,|\mathrm{I}|\;.$$

Next, we remove at most $r_{0,0}$ intervals I_1 from $\mathcal{I}_1$. Note that we do not specify which intervals should be removed but just give an upper bound on the number of intervals to be removed. Denote by $\mathcal{J}_1$ the resulting collection. Thus,

$$\#\mathcal{J}_1 \geqslant \#\mathcal{I}_1 - r_{0,0}\,. \tag{6}$$

For obvious reasons, intervals in $\mathcal{J}_1$ will be referred to as (level one) survivors. It will be convenient to define $\mathcal{J}_0 := \{I\}$. In general, for $n \geqslant 0$, given a collection $\mathcal{J}_n$ we construct a nested collection $\mathcal{J}_{n+1}$ of closed intervals J_{n+1} using the following two operations.

Splitting procedure. We subdivide each interval $J_n \in \mathcal{J}_n$ into R_n closed sub-intervals I_{n+1} of equal length and denote by $\mathcal{I}_{n+1}$ the collection of such intervals. Thus,

$$\#\mathcal{I}_{n+1} = R_n \times \#\mathcal{J}_n \qquad \text{and} \qquad |I_{n+1}| = R_n^{-1}\,|J_n|\;.$$

Removing procedure. For each interval $J_n \in \mathcal{J}_n$ we remove at most $r_{n,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within J_n. Note that the number of intervals I_{n+1} removed is allowed to vary amongst the intervals in $\mathcal{J}_n$. Next, for each interval $J_{n-1} \in \mathcal{J}_{n-1}$ we additionally remove at most $r_{n-1,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within J_{n-1}. In general, for each interval $J_{n-k} \in \mathcal{J}_{n-k}$ $(1 \leqslant k \leqslant n)$ we additionally remove at most $r_{n-k,n}$ intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that lie within J_{n-k}. Then the collection $\mathcal{J}_{n+1}$ consists of all intervals $I_{n+1} \in \mathcal{I}_{n+1}$ that survive after all these removing procedures for $k = 1, 2, \ldots, n$. Thus, the total number of survivors is at most

$$\#\mathcal{J}_{n+1} \geqslant R_n \#\mathcal{J}_n - \sum_{k=0}^{n} r_{k,n} \#\mathcal{J}_k. \tag{7}$$

Finally, having constructed the nested collections $\mathcal{J}_n$ of closed intervals we consider the limit set

$$\mathbf{K}(I, \mathbf{R}, \mathbf{r}) := \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J.$$

Any set $\mathbf{K}(I, \mathbf{R}, \mathbf{r})$ which can be achieved by the procedure described will be referred to as a $(I, \mathbf{R}, \mathbf{r})$ *Cantor set.*

Of course in general it can happen that for some choice of parameters $\mathbf{R}$ and $\mathbf{r}$ and some choice of removed intervals in removing procedure the $(I, \mathbf{R}, \mathbf{r})$ Cantor set becomes empty. However the next result shows that with some additional conditions on the parameters the Hausdorff dimension of this set is bounded below.

Theorem (BV4) *Given a* $(\mathrm{I}, \mathbf{R}, \mathbf{r})$ *Cantor set* $\mathbf{K}(I, \mathbf{R}, \mathbf{r})$*, suppose that* $R_n \geqslant 4$ *for all* $n \in \mathbb{Z}_{\geqslant 0}$ *and that*

$$\sum_{k=0}^{n} \left(r_{n-k,n} \prod_{i=1}^{k} \left(\frac{4}{R_{n-i}} \right) \right) \leqslant \frac{R_n}{4}. \tag{8}$$

Then
$$\dim \mathbf{K}(I,\mathbf{R},\mathbf{r}) \geqslant \liminf_{n\to\infty}(1-\log_{R_n} 2).$$

Here we use the convention that the product term in (8) is one when $k=0$ and by definition $\log_{R_n} 2 := \log 2/\log R_n$. The proof of Theorem BV4 is presented in [2, Theorem 4].

2.2 Duality between points and lines

The next two propositions show that there is a 'kind' of duality between rational points $P(p,r,q)$ and lines $L(A,B,C)$. It will play a crucial role in our proof.

Proposition 1 *Let $P_1(p_1,r_1,q_1), P_2(p_2,r_2,q_2)$ be two different rational points with $p_1/q_1 \neq p_2/q_2, r_1/q_1 \neq r_2/q_2$ and $0 < q_1||q_2\theta|| \leqslant q_2||q_1\theta||$. Let $L(A,B,C)$ with (A,B,C) satisfying (5) be the line passing through P_1,P_2. Assume that $(P_2)_\theta \in \Delta(P_1,\delta)$. Then*
$$(P_2)_\theta \in \Delta\left(L, \frac{\delta^2|B|}{q_2||q_1\theta||}\cdot\frac{H(P_2)}{H(P_1)}\right) \subset \Delta\left(L, 2\delta^2\frac{H(P_2)}{H(P_1)}\right). \tag{9}$$
Moreover,
$$H(L) \leqslant 4\delta H(P_1)\frac{q_2^3}{q_1^3}. \tag{10}$$

Proposition 2 *Let $L_1(A_1,B_1,C_1), L_2(A_2,B_2,C_2)$ be two lines with integer coefficients (A_i,B_i,C_i) satisfying (5) and $|A_2B_1| \leqslant |A_1B_2|$. Assume that they intersect at a rational point $P(p,r,q)$ and that $L_2\cap \mathrm{L}_\theta \in \Delta(L_1,\delta)$. Then*
$$L_2\cap \mathrm{L}_\theta \in \Delta\left(P, \frac{\delta^2 q}{|B_2A_1|}\cdot\frac{H(L_2)}{H(L_1)}\right) \subset \Delta\left(P, 2\delta^2\frac{H(L_2)}{H(L_1)}\right). \tag{11}$$
Moreover,
$$H(P) \leqslant 4\delta H(L_1)\frac{|B_2|^3}{|B_1|^3}. \tag{12}$$

Proof of Proposition 1. Since $P_1,P_2\in L$ we have the following system of equations
$$\begin{cases} Ap_1 - Br_1 + Cq_1 = 0; \\ Ap_2 - Br_2 + Cq_2 = 0; \\ A\theta - B\omega + C = 0 \end{cases}$$

where $\omega := \frac{A\theta+C}{B}$. Since $p_1/q_1 \neq p_2/q_2$ and $r_1/q_1 \neq r_2/q_1$ we get that the coefficients A and B are nonzero. Let $A' := A/d, B' := B/d$ where $d := (A,B)$. Then by $(A,B,C) = 1$ we get that $q_1 = dq_1'$ and $q_2 = dq_2'$. Then the first two equations of the system lead to
$$A'(p_1q_2' - p_2q_1') = B'(r_1q_2' - r_2q_1').$$
This together with $(A',B') = 1$ implies $|p_1q_2'-p_2q_1'| \geqslant |B'|$ and $|r_1q_2'-r_2q_1'| \geqslant |A'|$ or
$$\left|\frac{p_1}{q_1}-\frac{p_2}{q_2}\right| \geqslant \frac{|B|}{q_1q_2}, \quad \text{and} \quad \left|\frac{r_1}{q_1}-\frac{r_2}{q_2}\right| \geqslant \frac{|A|}{q_1q_2}.$$
The system also gives us the following equalities
$$|A|\left|\frac{p_1}{q_1}-\theta\right| = |B|\left|\frac{r_1}{q_1}-\omega\right| \quad \text{and} \quad |A|\left|\frac{p_2}{q_2}-\theta\right| = |B|\left|\frac{r_2}{q_2}-\omega\right|.$$

The assumption $(P_2)_\theta \in \Delta(P_1, \delta)$ is equivalent to

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| < \frac{\delta}{q_1^2||q_1\theta||}.$$

Finally by the triangle inequality we find that

$$\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \leqslant 2\max\left\{\left|\frac{p_1}{q_1} - \theta\right|, \left|\frac{p_2}{q_2} - \theta\right|\right\} = 2\max\left\{\frac{||q_1\theta||}{q_1}, \frac{q_2||\theta||}{q_2}\right\}.$$

By combining all these inequalities together we get that

$$|B| \leqslant q_1q_2\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \leqslant 2\max\{q_2||q_1\theta||, q_1||q_2\theta||\} = 2q_2||q_1\theta||;$$

$$|A| \leqslant q_1q_2\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| < \frac{\delta q_2}{q_1||q_1\theta||}.$$

Now we are ready to calculate the bound

$$\left|\frac{r_2}{q_2} - \omega\right| = \frac{|A|}{|B|}\frac{||q_2\theta||}{q_2} < \frac{1}{|AB|}\cdot\frac{\delta^2q_2^2\cdot||q_2\theta||}{q_1^2||q_1\theta||^2\cdot q_2} = \frac{1}{|A|B^2}\cdot\frac{\delta^2|B|\cdot H(P_1)}{q_2||q_1\theta||\cdot H(P_2)}.$$

Then the first inclusion in (9) follows immediately. For the second one we just use calculated estimate for $|B|$. Also by combining the bounds for $|A|$ and $|B|$ we get an estimate for the height $H(L)$:

$$H(L) = |A|B^2 \leqslant \frac{4\delta q_2^3||q_1\theta||}{q_1} = 4\delta H(P_1)\frac{q_2^3}{q_1^3}.$$

This completes the proof of Proposition 1.

⊠

Before we start the proof of Proposition 2 let's establish some basic facts regarding the point of intersection of two lines $L_1(A_1, B_1, C_1), L_2(A_2, B_2, C_2)$ with integer coefficients $(A_i, B_i, C_i) \in \mathbb{Z}^3\backslash(\{0\}^2 \times \mathbb{Z}), (A_i, B_i, C_i) = 1;\ i = 1, 2$. These facts will be of use in further discussion as well. An intersection $L_1 \cap L_2$ is a rational point $P(p, r, q)$ which is the solution of the following system of equations

$$\begin{cases} A_1p - B_1r + C_1q = 0; \\ A_2p - B_2r + C_2q = 0 \end{cases}$$

which leads to the following equalities

$$\frac{p}{q} = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1} \qquad \text{and} \qquad \frac{r}{q} = \frac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1}.$$

Therefore we get that

$$|B_1C_2 - B_2C_1| = dp,\ |A_1C_2 - A_2C_1| = dr,\ |A_1B_2 - A_2B_1| = dq. \tag{13}$$

where $d := \gcd(A_1B_2 - A_2B_1, B_1C_2 - B_2C_1) \in \mathbb{Z}$.

Let $i \in \{1, 2\}$. It is easily verified that

$$L_i \cap \mathrm{L}_\theta = \left(\theta, \frac{A_i\theta + C_i}{B_i}\right) = \left(\theta, \frac{r}{q} + \frac{A_i}{B_i}\left(\theta - \frac{p}{q}\right)\right).$$

Therefore

$$|L_1 \cap \mathrm{L}_\theta - L_2 \cap \mathrm{L}_\theta| = \left|\frac{A_1}{B_1} - \frac{A_2}{B_2}\right| \cdot \left|\theta - \frac{p}{q}\right| = \frac{d|q\theta - p|}{|B_1 B_2|}.$$

Hence

$$|q\theta - p| = d^{-1}|B_1 B_2| \cdot |L_1 \cap \mathrm{L}_\theta - L_2 \cap \mathrm{L}_\theta| \tag{14}$$

and

$$|q\omega - r| = \frac{|A_1|}{|B_1|}|q\theta - p|, \quad \text{where } \ \omega := \frac{A_1\theta + C_1}{B_1}. \tag{15}$$

Proof of Proposition 2. By (13) an upper bound for q is given by

$$q = d^{-1}|A_1B_2 - A_2B_1| \leqslant 2d^{-1}\max\{|A_1B_2|, |A_2B_1|\} = 2d^{-1}|A_1B_2|.$$

An upper bound for $|q\theta - p|$ can be derived from (14) and the assumption $L_2 \cap \mathrm{L}_\theta \in \Delta(P, \delta)$:

$$|q\theta - p| < \frac{\delta|B_1B_2|}{d|A_1|B_1^2} = \frac{\delta|B_2|}{d|A_1B_1|}$$

Finally we get the required bounds

$$|L_2 \cap \mathrm{L}_\theta - P_\theta| = \frac{|A_2|}{|B_2|} \cdot \frac{|q\theta - p|}{q} < \frac{|A_2| \cdot \delta^2|B_2|^2}{|B_2| \cdot d^2|A_1B_1|^2 \cdot q|q\theta - p|} \leqslant \frac{1}{q^2||q\theta||} \cdot \frac{\delta^2 q \cdot H(L_2)}{|B_2A_1| \cdot H(L_1)}$$

and

$$H(P) = q^2|q\theta - p| < 4d^{-2}|A_1B_2|^2 \cdot \frac{\delta|B_2|}{d|A_1B_1|} \leqslant 4\delta H(L_1) \cdot \frac{|B_2|^3}{|B_1|^3}.$$

To get the last inclusion in (11) we just use calculated bound for q. This completes the proof of Proposition 2.

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As we will see the duality between points and lines will appear throughout the whole paper.

3 Proof of Theorem 2

3.1 The idea

By definition for $\theta \in \mathbf{Bad}$ there exists a quantity $c(\theta) > 0$ such that

$$\inf_{q \in \mathbb{N}} q||q\theta|| = c(\theta).$$

In other words, for any positive integer q the following inequality is satisfied

$$q|q\theta - p| \geqslant c(\theta). \tag{16}$$

Let $R \geqslant e^9 c^{-1}(\theta)$ be an integer. Choose constants c and c_1 sufficiently small such that they satisfy the following inequalities

$$2^{12}c < 1, \quad 2c < R^2 c_1 c(\theta), \quad c < c(\theta) \tag{17}$$

and

$$2^6 \max\left\{\frac{c}{R^2 c_1 c(\theta)}, 2^{11}c\right\} \frac{(\log R + 2)^2 R^4}{(\log 2)^2} + 2^{15}c_1 \frac{R^3(\log R + 2)}{\log 2} < 1. \tag{18}$$

Finally choose the parameter $Q := c(\theta)R^2F(2)$ where

$$F(n) := \prod_{k=1}^{n} k\,[\log^* k] \ \text{ for } n \geqslant 1 \qquad \text{and } F(n) := 1 \ \text{ for } n \leqslant 0.$$

The goal is to construct a $(I, \mathbf{R}, \mathbf{r})$ Cantor type set $\mathbf{K}_c$ with properly chosen parameters $I, \mathbf{R}$ and $\mathbf{r}$ so that $\mathbf{K}_c$ is a subset of $\mathbf{Mad}_P(f, c, Q) \cap \mathbf{Mad}_L(f, c, Q)$. Then we use Theorem BV4 to estimate its Hausdorff dimension. Let I be any interval of length c_1 contained within the unit interval $\{\theta\} \times [0, 1] \subset \mathrm{L}_\theta$. Define $\mathcal{J}_0 := \{I\}$. We are going to construct, by induction on n, a collection $\mathcal{J}_n$ of closed intervals J_n such that $\mathcal{J}_n$ is nested in $\mathcal{J}_{n-1}$; that is, each interval J_n in $\mathcal{J}_n$ is contained in some interval J_{n-1} in $\mathcal{J}_{n-1}$. The length of an interval J_n will be given by

$$|J_n| \;:=\; c_1\, R^{-n} F^{-1}(n).$$

Moreover, each interval J_n in $\mathcal{J}_n$ will satisfy the conditions that

$$\begin{aligned} J_n \cap \Delta(P, cf^{-1}(q)) \;=\; \emptyset \quad \forall \;\; & P(p, r, q) \in \mathbb{Q}^2 \ \text{ with } (p, r, q) = 1, \\ & Q < H(P) < c(\theta)R^{n-1}F(n-1) \end{aligned} \tag{19}$$

and

$$\begin{aligned} J_n \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \;=\; \emptyset \quad & \forall\ L(A, B, C) \text{ with } (A, B, C) \in \mathbb{Z}^3,\ B \neq 0, \\ & (A, B, C) = 1,\ Q < H(L) < c(\theta)R^{n-1}F(n-1) \end{aligned} \tag{20}$$

In particular, we put

$$\mathbf{K}_c = \bigcap_{n=1}^{\infty} \bigcup_{J \in \mathcal{J}_n} J \,.$$

By construction, conditions (19) and (20) ensure that

$$\mathbf{K}_c \subset \mathbf{Mad}_P(f, c) \cap \mathbf{Mad}_L(f, c) \cap \mathrm{L}_\theta \,.$$

The aim of the rest of the paper is to show that $\mathbf{K}_c$ is in fact a $(I, \mathbf{R}, \mathbf{r})$ Cantor set with $\mathbf{R} = (R_n)$ given by

$$R_n := R\,(n+1)\,[\log^*(n+1)] \tag{21}$$

and $\mathbf{r} = (r_{m,n})$ given by

$$r_{m,n} \;:=\; \begin{cases} 25R\log R \cdot n^4(\log^* n)^4 & \text{if } \ m = n-3 \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Then Theorem 2 will follow from Theorem BV4. Indeed for $n < 3$ the condition (8) is obviously satisfied. For $n \geqslant 3$ and $R \geqslant 2^7$ we have that the

$$\begin{aligned} \text{l.h.s. of (8)} \quad &= \quad r_{n-3,n} \cdot \frac{4^3}{R_{n-1}R_{n-2}R_{n-3}} \\ &\leqslant \quad \frac{4^3}{R^3} \cdot \frac{25R\log R \cdot n^4(\log^* n)^4}{n(n-1)(n-2)\log^* n \log^*(n-1)\log^*(n-2)} \\ &\leqslant \quad \frac{16 \cdot 25 \cdot 4^3 \log R}{R^3} \cdot \frac{R(n+1)[\log^*(n+1)]}{4} \leqslant \frac{R_n}{4} \quad = \ \text{r.h.s. of (8)}\,. \end{aligned}$$

Therefore Theorem BV4 implies that

$$\dim \mathbf{K}_c \geqslant \liminf_{n\to\infty}(1 - \log_{R_n} 2) = 1$$

which completes the proof of Theorem 2.

3.2 Basic construction. Splitting into collections $C_P(n,l,k)$ and $C_L(n,l,k)$

Now we will describe the procedure of constructing the collections $\mathcal{J}_n$. For $n=0$, we trivially have that (19), (20) are satisfied for the sole interval $I \in \mathcal{J}_0$. The point is that by the choice of Q there are neither points nor lines satisfying the height condition $Q < H(P), H(L) < c(\theta)$. Then we construct $\mathcal{J}_i, i = 1,2,3$ by just subdividing each J_{i-1} in $\mathcal{J}_{i-1}$ into $R \cdot i[\log^* i]$ closed intervals of equal length. Again for the same reason the conditions (19) and (20) are satisfied for any $J_i \in \mathcal{J}_i, i = 1,2,3$. Note that

$$\#\mathcal{J}_i = R^i F(i), \quad i = 1,2,3.$$

In general, given $\mathcal{J}_n$ satisfying (19) and (20) we wish to construct a nested collection $\mathcal{J}_{n+1}$ of intervals J_{n+1} for which (19) and (20) are satisfied with n replaced by $n+1$. By definition, any interval J_n in $\mathcal{J}_n$ avoids intervals $\Delta(P, cf^{-1}(q))$ and $\Delta(L, cf^{-1}(|A|^*|B|^*))$ arising from points and lines with height bounded above by $c(\theta)R^{n-1}F(n-1)$. Since any 'new' interval J_{n+1} is to be nested in some J_n, it is enough to show that J_{n+1} avoids intervals $\Delta(P, cf^{-1}(q))$ and $\Delta(L, cf^{-1}(|A|^*|B|^*))$ arising from points and lines with height satisfying

$$c(\theta)R^{n-1}F(n-1) \leqslant H(P), H(L) < c(\theta)R^n F(n) \ . \tag{23}$$

Denote by $C_P(n)$ the collection of all rational points satisfying this height condition. Formally

$$C_P(n) := \left\{ P(p,r,q) \in \mathbb{Q}^2 \ : \ P \ \text{ satsifies } (23) \right\}$$

and it is precisely this collection of rationals that comes into play when constructing $\mathcal{J}_{n+1}$ from $\mathcal{J}_n$. By analogy for 'lines' let

$$C_L(n) := \{ L(A,B,C) \ : \ L \ \text{ satsifies } (23) \} \ .$$

We now proceed with the construction. Assume that $n \geqslant 3$. We subdivide each J_n in $\mathcal{J}_n$ into $R_n = [R(n+1)\log^*(n+1)]$ closed intervals I_{n+1} of length

$$|I_{n+1}| = c_1 R^{-n-1} F^{-1}(n+1).$$

Denote by $\mathcal{I}_{n+1}$ the collection of such intervals. In view of the nested requirement, the collection $\mathcal{J}_{n+1}$ which we are attempting to construct will be a sub-collection of $\mathcal{I}_{n+1}$. In other words, the intervals I_{n+1} represent possible candidates for J_{n+1}. The goal now is simple — it is to remove those 'bad' intervals I_{n+1} from $\mathcal{I}_{n+1}$ for which

$$I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \quad \text{for some } \ P(p,r,q) \in C_P(n) \tag{24}$$

or

$$I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset \quad \text{for some } \ L(A,B,C) \in C_L(n) \ . \tag{25}$$

So we define

$$\mathcal{J}_{n+1} := \left\{ J_{n+1} \in \mathcal{I}_{n+1} \ : \ \begin{array}{l} J_{n+1} \cap \Delta(P, cf^{-1}(q)) = \emptyset \ \text{ for any } P \in C_P(n) \\ J_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) = \emptyset \ \text{ for any } L \in C_L(n). \end{array} \right\}$$

Consider the rational point $P(p,r,q) \in C_P(n)$. Note that since $q^2 \geqslant q^2||q\theta|| = H(q) \geqslant cR^{n-1}F(n-1)$, we have that

$$f(q) \geqslant \frac{1}{2}\log^*(cR^{n-1}F(n-1))\log^*\frac{1}{2}\log(c(\theta)R^{n-1}F(n-1)) > \frac{1}{2}n(\log^* n)^2 \tag{26}$$

for sufficiently large R. We use Stirling formula to show that for $n \geqslant 3$,

$$c(\theta)R^{n-1}F(n-1) \geqslant c(\theta)R^{n-1}(n-1)! > (8n)^n \quad \text{for } R \geqslant e^9 c^{-1}(\theta).$$

Therefore the left hand side of (26) is bigger than

$$\frac{1}{2}n\log(8n) \cdot \log^*(\frac{1}{2}n\log(8n)) > \frac{1}{2}n\log^{*2} n.$$

Note that for any line $L(A,B,C) \in C_L(n)$ we have the analogous bound

$$f(|A|^*|B|^*) \geqslant \frac{1}{2}n(\log^* n)^2. \tag{27}$$

For $l \in \mathbb{Z}$ we split $C_P(n)$ into sub-collections

$$C_P(n,l) := \left\{ P(p,r,q) \in C_P(n) : \begin{array}{c} c(\theta)2^l R^{n-1}F(n-1) \leqslant H(P) \\ H(P) < c(\theta)2^{l+1}R^{n-1}F(n-1) \end{array} \right\}. \tag{28}$$

In view of (23) we have that

$$2^l < Rn\log^* n \tag{29}$$

so

$$0 \leqslant l \leqslant [\log_2(Rn\log^* n)] < \log_2 R + 2\log_2 n < c_3 \log^* n. \tag{30}$$

where $c_3 := (\log R + 2)/\log 2$ is an absolute constant independent on n and l.

Additionally with $k \in \mathbb{Z}$ we split the collection $C_P(n,l)$ into sub-collections $C'_P(n,l,k)$ such that

$$C'_P(n,l,k) := \left\{ P(p,r,q) \in C_P(n,l) \ : \ c(\theta)2^k \leqslant q||q\theta|| < c(\theta)2^{k+1} \right\}. \tag{31}$$

Take any $P(p,r,q) \in C'_P(n,l,k)$. In view of (16) the value k should be nonnegative. On the other hand one can get an upper bound for k by (23):

$$0 \leqslant k \leqslant [\log_2(R^n F(n))] < n\log_2 R + n\log_2 n + n\log_2 \log^* n < c_3 n\log^* n, \tag{32}$$

The upshot is that for fixed n, l the number of classes $C'_P(n,l,k)$ is at most $c_3 n\log^* n$.

Note that within the collection $C'_P(n,l,k)$ we have very sharp control of the height $H(P)$. Then by (28) and (31) we also have very sharp control on the value q as well, namely

$$2^{l-k-1}R^{n-1}F(n-1) < q < 2^{l-k+1}R^{n-1}F(n-1). \tag{33}$$

Concerning the collection $C_L(n)$ we also partition it into sub-collections. Firstly we partition it into sub-collections $C_L(n,l)$ such that

$$C_L(n,l) := \left\{ L \in C_L(n) : \begin{array}{c} c(\theta)2^l R^{n-1}F(n-1) \leqslant H(L) \\ H(L) < c(\theta)2^{l+1}R^{n-1}F(n-1) \end{array} \right\}. \tag{34}$$

Then we split $C_L(n,l)$ into sub-collections $C'_L(n,l,k)$ such that

$$C'_L(n,l,k) := \{L(A,B,C) \in C_L(n,l) \ : \ 2^k \leqslant |B| < 2^{k+1}\} \tag{35}$$

One can check that l and k satisfy the same conditions (30) and (32) as in the case of points. Note that within each collection we have very good control of all point and line parameters.

The procedure of removing "bad" intervals from $\mathcal{I}_{n+1}$ will be as follows. We will firstly remove all intervals $I_{n+1} \in \mathcal{I}_{n+1}$ such that there exists a point $P \in C_P(n,0)$ which satisfy $I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset$ or there exists a line a line $L \in C_L(n,0)$ which satisfy $I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset$. Then we repeat this removing procedure for collections

$$C_P(n,1) \text{ and } C_L(n,1), \ldots,\ C_P(n, c_2 \log^* n) \text{ and } C_L(n, c_2 \log^* n)$$

in exactly this order.

We will use lexicographical order for pairs in $\mathbb{Z}^2$. That is, we say that $(a,b) \leqslant (c,d)$ if either $a < c$ or $a = c, b \leqslant d$. Consider the point $P(p,r,q) \in C'_P(n,l,k)$. If there exists a pair $(n',l') \leqslant (n,l)$ and a line $L(A,B,C) \in C_L(n',l')$ such that

$$H(L) < H(P) \quad \text{and} \quad \Delta(P, cf^{-1}(q)) \subset \Delta(L, cf^{-1}(|A|^*|B|^*))$$

then such a point will not remove anything more than was removed by a line L. Therefore such a point can be ignored. The same is true if there exists a point $P'(p',r',q') \in C_P(n',l')$ such that

$$H(P') < H(P) \quad \text{and} \quad \Delta(P, cf^{-1}(q)) \subset \Delta(P', cf^{-1}(q')).$$

Therefore instead of collection $C'_P(n,l,k)$ we can work with

$$C_P(n,l,k) := \left\{ P(p,r,q) \in C'_P(n,l,k) \;\middle|\; \begin{array}{l} \forall (n',l') < (n,l), \\ \forall L(A,B,C) \in C_L(n',l') \text{ with } H(L) < H(P), \\ \forall P'(p',r',q') \in C_P(n',l') \text{ with } H(P') < H(P) \\ \Delta(P, cf^{-1}(q)) \not\subset \Delta(L, cf^{-1}(|A|^*|B|^*)), \\ \Delta(P, cf^{-1}(q)) \not\subset \Delta(P', cf^{-1}(q')). \end{array} \right\}$$

By the same procedure we construct the collection $C_L(n,l,k)$ from $C'_L(n,l,k)$. Note that by the construction of $C_P(n,l,k)$ there exists at most one point $P(p,r,q) \in C_P(n,l,k)$ with given second coordinate r/q.

3.3 Blocks of intervals $\mathbf{B}_P(J)$ and $\mathbf{B}_L(J)$

Take the maximal possible constant $c_2 > 0$ such that

$$c_2 \leqslant \frac{1}{2^{10} c(\theta)} \quad \text{and} \quad \frac{R^2 c_1}{c_2} \in \mathbb{Z}. \tag{36}$$

Fix the triple (n,l,k) and consider an arbitrary interval $J \subset \mathrm{L}_\theta$ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Then for any $P(p,r,q) \in C_P(n,l,k)$ we have $|\Delta(P, cf^{-1}(q))| < |J|$. Indeed this is true because

$$|J| \geqslant |\Delta(P, cf^{-1}(q))| \Leftrightarrow \frac{c_2}{2^l R^{n-1} F(n-1)} \geqslant \frac{2c}{f(q) H(P)}$$

$$\overset{(28)}{\Leftarrow} \frac{c_2}{2^l R^{n-1} F(n-1)} \geqslant \frac{2c}{c(\theta) 2^l R^{n-1} F(n-1) \cdot f(q)}.$$

The last inequality is true provided $c_2 c(\theta) \geqslant 2c$ which in turn is true by the second inequality of (17) and (36). One can easily check that the same fact is true for any $\Delta(L, cf^{-1}(|A|^*|B|^*))$ where $L(A,B,C) \in C_L(n,l,k)$.

Lemma 1 *Let J be an interval on L_θ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Then all rational points $P(p,r,q) \in C_P(n,l,k)$ such that $\Delta(P, cf^{-1}(q)) \cap J \neq \emptyset$ lie on a single line.*

Proof. Consider an arbitrary point $P(p,r,q)\in C_P(n,l,k)$. Then

$$\left|\theta-\frac{p}{q}\right| = \frac{H(P)}{q^3} \overset{(28),(33)}{<} \frac{c(\theta)}{2^{2l-3k-4}R^{2(n-1)}F^2(n-1)}. \tag{37}$$

Suppose we have three points $P_i(p_i,r_i,q_i)\in C_P(n,l,k), i=1,2,3$ such that $\Delta(P_i, cf^{-1}(q_i))\cap J\neq\emptyset$ and they do not lie on a single line. Then they form a triangle which has the area at least

$$\mathbf{area}(\triangle P_1P_2P_3)\geqslant \frac{1}{2q_1q_2q_3} \overset{(33)}{\geqslant} \frac{1}{2^{3l-3k+4}R^{3(n-1)}F^3(n-1)}.$$

On the other hand the first coordinates p_i/q_i of the points P_i should satisfy (37) and their second coordinates r_i/q_i should lie within the interval of length $|J|+|\Delta(P_i,cf^{-1}(q_i))|\leqslant 2|J|$. Therefore we have the following upper bound for the area of triangle $\triangle P_1P_2P_3$:

$$\mathbf{area}(\triangle P_1P_2P_3) < \frac{2c_2 2^{-l}R^{-n+1}F^{-1}(n-1)\cdot 2c(\theta)}{2^{2l-3k-4}R^{2(n-1)}F^2(n-1)}$$

$$\leqslant 2^{10}c_2c(\theta)\cdot\frac{1}{2^{3l-3k+4}R^{3(n-1)}F^3(n-1)}.$$

Finally by (36) we get that the last value is bounded above by

$$\frac{1}{2^{3l-3k+4}R^{3(n-1)}F^3(n-1)}\leqslant \mathbf{area}(\triangle P_1P_2P_3)$$

which is impossible. So we get a contradiction.

⊠

So given interval J of length $c_2 2^{-l}R^{n-1}F^{-1}(n-1)$ if we have at least two points $P\in C_P(n,l,k)$ as in Lemma 1 then all the points with such property will lie on a single line L. We denote this line by L_J. If there is at most one point $P\in C_p(n,l,k)$ as in Lemma 1 then we just say that L_J is undefined.

Note that L_J can not be horizontal because by the construction of $C_P(n,l,k)$ there is only one point $P(p,r,q)\in C_P(n,l,k)$ with given second coordinate r/q. L_J can not be vertical too. Otherwise its equation can be written as $x=C/A$, $\gcd(A,C)=1$. Then by the construction of θ we have that

$$\left|\theta-\frac{p}{q}\right| = \left|\theta-\frac{C}{A}\right|\geqslant\frac{c(\theta)}{A^2}$$

which together with (37) gives us

$$|A|\geqslant 2^{l-3/2k-2}R^{n-1}F(n-1).$$

Then by defitnition of L_J there exist two points $P_1(p_1,r_1,q_1), P_2(p_2,r_2,q_2)$ with $|r_1/q_1-r_2/q_2|<2|J|$. However

$$\left|\frac{r_1}{q_1}-\frac{r_2}{q_2}\right|\geqslant\frac{|A|}{q_1q_2}\overset{(33)}{\geqslant} 2^{-l+k/2-4}R^{-n+1}F^{-1}(n-1)>2|J|.$$

So we get a contradiction.

The statement of Lemma 1 can be strengthened if we have more than two points $P\in C_P(n,l,k)$ such that $\Delta(P,cf^{-1}(q))\cap J\neq\emptyset$.

Lemma 2 *Let J be an interval on L_θ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Assume that there exists a line L_J. Consider the sequence of consecutive intervals $M_i \subset \mathrm{L}_\theta$, $i \in \mathbb{N}$, $|M_i| = |J|$, $M_1 := J$ and bottom end of M_i coincides with the top end of M_{i+1}. Define the set*

$$\mathcal{P}(J,m) := \left\{ P \in C_P(n,l,k) \ : \ P \in L_J \text{ and } \Delta(P, cf^{-1}(q)) \cap \left(\bigcup_{i=1}^{m} M_i \right) \neq \emptyset \right\}$$

and the value

$$m_P(J) := \max\{m \in \mathbb{N} \mid \#\mathcal{P}(J,m) \geqslant m+1\}.$$

Then all rational points $P \in C_P(n,l,k)$ such that

$$\Delta(P, cf^{-1}(q)) \cap \left(\bigcup_{i=1}^{m_P(J)} M_i \right) \neq \emptyset$$

lie on a line L_J.

Remark 1. Since the number of points $P \in C_P(n,l,k), P \in L_J$ is finite, the value $m_P(J)$ is correctly defined. Indeed since by assumption $\#\mathcal{P}(J,1) \geqslant 2$, $m+1 \to \infty$ and $\#\mathcal{P}(J,m)$ is bounded then $m(J)$ exists and is finite.

Remark 2. We define the block of intervals

$$\mathbf{B}_P(J) := \bigcup_{i=1}^{m(J)} M_i.$$

We will work with it as with one unit. If for some interval J the line L_J is undefined then we define $m(J) := 1$ and $\mathbf{B}_P(J) := J$. So now $m(J)$ and $\mathbf{B}_P(J)$ are well defined for all intervals J of length $c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$.

Proof. Is similar to the proof of Lemma 1. Let

$$\mathcal{P}(J, m(J)) = (P_i(p_i, r_i, q_i))_{1 \leqslant i \leqslant m(J)+1}$$

where the sequence r_i/q_i is ordered in ascending order. Assume that there is a point $P(p,r,q) \in C_P(n,l,k)$ such that $P \notin L_J$ and $\Delta(P, cf^{-1}(q)) \cap \mathbf{B}_P(J) \neq \emptyset$. Then the triangle $\Delta(PP_1P_{m(J)+1})$ is splitted into $m_P(J)$ disjoint triangles

$$\Delta(PP_iP_{i+1}), \quad 1 \leqslant i \leqslant m_P(J)$$

each of which has the area

$$\mathbf{area}(\Delta(PP_iP_{i+1})) \geqslant \frac{1}{2qq_iq_{i+1}}.$$

On the other hand the first coordinates of the points $P_1, \ldots, P_{m_P(J)+1}$ and P satisfy (37) and their second coordinates lie within the interval of length at most $(m_P(J)+1)|J|$. Therefore we have the following estimate for the area of the triangle

$$\frac{m_P(J)}{2^{3l-3k+4} R^{3(n-1)} F^3(n-1)} \leqslant \mathbf{area}(\triangle PP_1P_{m_P(J)+1}) \leqslant \frac{2^9(m_P(J)+1)c_2c(\theta)}{2^{3l-3k+4} R^{3(n-1)} F^3(n-1)}.$$

which is impossible since the l.h.s of this inequality is bigger than its r.h.s.

$\boxtimes$

Lemmas 1 and 2 have their full analogues for lines $L \in C_L(n,l,k)$. However the proofs areslightly different. We will formulate them in the next two lemmata.

Lemma 3 *Let J be an interval on L_θ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Then all lines $L(A,B,C) \in C_L(n,l,k)$ such that $\Delta(L, cf^{-1}(|A|^*|B|^*)) \cap J \neq \emptyset$ pass through a single rational point P.*

Proof. We will use the following well-known fact. Let us have three planar lines $L_i(A_i, B_i, C_i), i = 1,2,3$ defined by equations $A_i x - B_i y + C_i = 0$. Then they intersect in one point (probably at infinity) if and only if

$$\det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = 0.$$

Suppose that there are three lines $L_1, L_2, L_3 \in C_L(n,l,k)$ which do not intersect at one point but their thickenings intersect J. Then

$$\left| \det \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \right| \geqslant 1.$$

On the other hand we can make a vertical shifts of L_1, L_2, L_3 to the distances $\delta_i < |J| + |\Delta(L_i)| < 2|J|$, $i = 1,2,3$ such that they will intersect at one point on J. By vertically shifting a line to the distance ϵ we change its C-coordinate by the value $B\epsilon$. Therefore we have

$$\det \begin{pmatrix} A_1 & B_1 & C_1 + B_1\delta_1 \\ A_2 & B_2 & C_2 + B_2\delta_2 \\ A_3 & B_3 & C_3 + B_3\delta_3 \end{pmatrix} = 0 \quad \Rightarrow \quad \left| \det \begin{pmatrix} A_1 & B_1 & B_1\delta_1 \\ A_2 & B_2 & B_2\delta_2 \\ A_3 & B_3 & B_3\delta_3 \end{pmatrix} \right| \geqslant 1.$$

However the latter determinant is bounded above by

$$2|J|(|B_1(A_2B_3 - A_3B_2)| + |B_2(A_1B_3 - A_3B_1)| + |B_3(A_1B_2 - A_2B_1)|)$$

$$\stackrel{(34),(35)}{\leqslant} 2c_2 c(\theta) 2^{-l} R^{-n+1} F^{-1}(n-1) \cdot 6 \cdot 2^{l+3} R^{n-1} F(n-1) \stackrel{(36)}{<} 1$$

We get a contradiction.

⊠

So given interval J of length $c_2 2^{-l} R^{n-1} F^{-1}(n-1)$ if we have at least two lines from $C_L(n,l,k)$ as in Lemma 3 then all lines with such property will intersect at one rational point P. We denote this point by P_J. If there is at most one line from $C_L(n,l,k)$ as in Lemma 3 then we just say that P_J is undefined.

The next Lemma is a "line" analogue of Lemma 2.

Lemma 4 *Let J be an interval on L_θ of length $|J| = c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$. Assume that there exists a point P_J. Consider the sequence of consecutive intervals $M_i \subset \mathrm{L}_\theta$, $i \in \mathbb{N}$, $|M_i| = |J|$, $M_1 := J$ and bottom end of M_i coincides with the top end of M_{i+1}. Define the set*

$$\mathcal{L}(J,m) := \left\{ L \in C_L(n,l,k) \ : \ P_J \in L \text{ and } \Delta(L, cf^{-1}(|A|^*|B|^*)) \cap \left(\bigcup_{i=1}^{m} M_i \right) \neq \emptyset \right\}$$

and the value

$$m_L(J) := \max\{m \in \mathbb{N} \mid \#\mathcal{L}(J,m) \geqslant m+1\}.$$

Then all lines $L \in C_L(n,l,k)$ such that

$$\Delta(L, cf^{-1}(|A|^*|B|^*)) \cap \left(\bigcup_{i=1}^{m_L(J)} M_i \right) \neq \emptyset$$

intersect at a point P_J.

By analogy with Remark 1 the value $m_L(J)$ is correctly defined. We define the block of intervals

$$\mathbf{B}_L(J) := \bigcup_{i=1}^{m_L(J)} M_i.$$

We will work with it as with one unit. As in Remark 2 if for some interval J the point P_J is not defined then we define $m_L(J) := 1$ and $\mathbf{B}_L(J) := J$.

Proof. If $m_L(J) = 1$ then this is simply the statement of Lemma 3. Now assume that $m_L(J) > 1$. Let

$$\mathcal{L}(J, m_L(J)) = (L_i(A_i, B_i, C_i))_{1 \leqslant i \leqslant m_L(J)+1}.$$

Denote by

$$\omega_i := \frac{A_i\theta + C_i}{B_i}, \quad 1 \leqslant i \leqslant m_L(J) + 1.$$

Then all the triples (A_i, B_i, C_i) lie inside the figure F defined by the inequalities

$$\begin{aligned} |A_i| = \frac{H(L_i)}{|B_i|^2} &\overset{(34),(35)}{<} c(\theta)2^{l-2k+1}R^{n-1}F(n-1), \\ |B_i| &\overset{(35)}{<} 2^{k+1} \quad \text{and} \\ |A_i\theta - B_i\omega_1 + C_i| < |B_i| \cdot |\omega_1 - \omega_i| &< c_2 m_L(J) 2^{k+2-l} R^{-n+1} F^{-1}(n-1). \end{aligned}$$

The volume of this figure is $16c_2c(\theta)m_L(J)$ which in view of (36) is smaller than $\frac{1}{6}m_L(J)$. All points (A_i, B_i, C_i) together with $(0,0,0)$ lie on the plane defined by $A_ip - B_ir + C_iq = 0$. And since $\gcd(A_i, B_i, C_i) = 1$ their convex body contains at least $m_L(J)$ disjoint triangles with vertices in points (A_i, B_i, C_i) and $(0,0,0)$.

Now suppose that there is a line $L(A,B,C) \in C_L(n,l,k)$ such that $P_J \notin L$ and $\Delta(L, cf^{-1}(|A|^*|B|^*)) \cap \mathbf{B}_L(J) \neq \emptyset$. Then $(A,B,C) \in F$ but now this point doesn't lie on the same plane with points (A_i, B_i, C_i) and $(0,0,0)$. Then it formes at least $m_L(J)$ disjoint tetrahedrons with them each of which has the volume at least $1/6$. Therefore the volume of F is bounded by

$$\frac{1}{6}m_L(J) \leqslant \mathbf{vol}(F) < \frac{1}{6}m_L(J).$$

But the last inequality is impossible. Therefore the line L has to pass through the point P_J.

$\boxtimes$

3.4 Properties of blocks $\mathbf{B}_P(J)$, $\mathbf{B}_L(j)$ and quantities m_P, m_L

Take an arbitrary interval M of length $c_2 2^{-l}R^{n-1}F(n-1)$ and consider the collection $\mathcal{P}_M$ of points $P \in C_P(n,l,k)$ such that $\Delta(P, cF^{-1}(q)) \cap \mathbf{B}_P(M) \neq \emptyset$. Then one of the following cases should be true.

Case 1P. For any interval $J \subset \mathbf{B}_P(M)$ such that $|J| = |M|$,

$$\#\mathcal{S}_J := \#\{P \in \mathcal{P}_M \mid \Delta(P, cF^{-1}(q)) \cap J \neq \emptyset\} \leqslant 2^2.$$

Case 2P. There exists $J \subset \mathbf{B}_P(M), |J| = |M|$ such that $\#\mathcal{S}_J > 2^2$. Then the line L_J is correctly defined and therefore $L_M = L_M(A, B, C)$ is correctly defined as well. Let the coefficient B satisfy the condition

$$|B| < \frac{c(\theta)2^{k+6}}{1/2cn(\log^* n)^2} \tag{38}$$

Case 3P. There exists $J \subset \mathbf{B}_P(M), |J| = |M|$ such that $\#\mathcal{S}_J > 2^2$ and

$$|B| \geqslant \frac{c(\theta)2^{k+6}}{1/2cn(\log^* n)^2}. \tag{39}$$

Consider Cases 2P and 3P. Since for any $P \in \mathcal{S}_J$ all numbers P_θ lie inside an interval of length at most $2|J|$ there are at least two points $P_1(p_1, r_1, q_1)$ and $P_2(p_2, r_2, q_2)$ from $\mathcal{S}_J$ such that

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| < 2^{-1}|J|.$$

Without loss of generality assume that $q_2||q_1\theta|| > q_1||q_2\theta||$. Then

$$(P_2)_\theta \in \Delta(P_1, 2^{-1}|J|H(P_1)) \overset{(28)}{\subset} \Delta(P_1, c(\theta)c_2) \overset{(36)}{\subset} \Delta(P_1, 2^{-10}). \tag{40}$$

Since L_M is neither vertical nor horizontal, Proposition 1 is applicable for $\delta = 2^{-10}$. It states that

$$(P_2)_\theta \in \Delta\left(L_M, \frac{2^{-20}|B|}{q_2||q_1\theta||} \cdot \frac{H(P_2)}{H(P_1)}\right)$$

and

$$H(L_M) \leqslant 2^{-8}H(P_1)\frac{q_2^3}{q_1^3} \overset{(33)}{\leqslant} \frac{1}{2}H(P_1).$$

It shows that L_M belongs to the class which within the basic construction had been considered before considering the points from $\mathcal{P}_M$.

Now let's consider the Case 2P. By (38), (28), (31) and (33) the inclusion (40) implies that

$$(P_2)_\theta \in \Delta\left(L_M, \frac{2^{-11}}{1/2cn(\log^* n)^2}\right) \overset{(27)}{\subset} \Delta\left(L_M, \frac{1}{8cf(|A|^*|B|^*)}\right).$$

Now since for any $P(p', r', q') \in C_P(n, l, k)$ the distance $|\theta - p'/q'|$ can differ from $|\theta - p/q|$ by factor at most 4 the same thing is true for the value $|\omega - r'/q'|$. An implication of this is that for all $P \in \mathcal{P}_M$,

$$P_\theta \in \Delta(L_M, 1/2cf^{-1}(|A|^*|B|^*)).$$

Whence

$$\bigcup_{P(p,r,q)\in\mathcal{P}_M} \Delta(P, cf^{-1}(q)) \subset \Delta(L_M, cf^{-1}(|A|^*|B|^*)).$$

However by the construction of the collection $C_P(n, l, k)$, for all $P \in C_P(n, l, k)$ intervals $\Delta(P, cf^{-1}(q))$ are not contained in any interval Δ previously considered. Therefore since $\mathcal{P}_M \subset C_P(n, l, k)$ then the set $\mathcal{P}_M$ in case 2P should be empty — a contradiction. Therefore the Case 2P is impossible.

Consider the last Case 3P. Let's order all the points in $\mathcal{P}_M = (P_i(p_i, r_i, q_i))_{1\leqslant i\leqslant m_L(J)+1}$ in such a way that the sequence p_i/q_i is increasing. Then we have

$$\left|\frac{p_1}{q_1} - \frac{p_{m_P(M)+1}}{q_{m_P(M)+1}}\right| \leqslant \left|\theta - \frac{p_1}{q_1}\right| + \left|\theta - \frac{p_{m_P(M)+1}}{q_{m_P(M)+1}}\right| \overset{(37)}{\leqslant} \frac{c(\theta)}{2^{2l-3k-5}R^{2(n-1)}F^2(n-1)}.$$

On the other hand the smallest possible difference between consecutive numbers p_i/q_i and p_{i+1}/q_{i+1} is bounded below by

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} \geqslant \frac{|B|}{q_i q_{i+1}}.$$

and therefore

$$\left|\frac{p_1}{q_1} - \frac{p_{m_P(M)+1}}{q_{m_P(M)+1}}\right| \overset{(33)}{\geqslant} \frac{|B| m_P(M)}{2^{2l-2k+2} R^{2(n-1)} F^2(n-1)}.$$

By combining the last two inequalities and (39) we finally get an estimate

$$m_P(M) \leqslant cn(\log^* n)^2. \tag{41}$$

Now for the same interval M define the collection $\mathcal{L}_M$ of lines $L(A, B, C) \in C_L(n, l, k)$ such that $\Delta(L, cF^{-1}(|A|^*|B|^*)) \cap \mathbf{B}_L(M) \neq \emptyset$. Consider three different cases which will be full analogues to cases 1P, 2P and 3P.

Case 1L. For any interval $J \subset \mathbf{B}_L(M)$ such that $|J| = |M|$,

$$\#\mathcal{S}_J := \#\{L(A, B, C) \in \mathcal{L}_M \mid \Delta(L, cF^{-1}(|A|^*|B|^*)) \cap J \neq \emptyset\} \leqslant 2^2.$$

Case 2L. There exists $J \subset \mathbf{B}_L(M)$, $|J| = |M|$ such that $\#\mathcal{S}_J > 2^2$. Then the point P_J is correctly defined and therefore $P_M = P_M(p, r, q)$ is correctly defined as well. Let the coefficient q satisfy the condition

$$q < \frac{c(\theta)2^{l-k+3} R^{n-1} F(n-1)}{1/2cn(\log^* n)^2}. \tag{42}$$

Case 3L. There exists $J \subset \mathbf{B}_L(M), |J| = |M|$ such that $\#\mathcal{S} > 2^2$ and

$$q \geqslant \frac{c(\theta)2^{l-k+3} R^{n-1} F(n-1)}{1/2cn(\log^* n)^2}. \tag{43}$$

Consider Cases 2L and 3L. The arguments will be essentially the same to that about Cases 2P and 3P. So one can get that there are at least two lines $L_1(A_1, B_1, C_1)$ and $L_2(A_2, B_2, C_2)$ from $\mathcal{S}$ such that

$$|L_1 \cap \mathrm{L}_\theta - L_2 \cap \mathrm{L}_\theta| < 2^{-1}|J|.$$

Without loss of generality suppose that $|A_2 B_1| < |A_1 B_2|$. Then

$$L_2 \cap \mathrm{L}_\theta \in \Delta(L_1, 2^{-10}).$$

and Proposition 2 is applicable with $\delta = 2^{-10}$. Therefore arguments analogous to those used in cases 2P, 3P give us

$$L_2 \cap \mathrm{L}_\theta \in \left(P_M, \frac{2^{-19} q}{|B_2 A_1|}\right)$$

and

$$H(P_M) \leqslant \frac{1}{2} H(L_1).$$

Therefore the point P_M is from the class which has already been considered before considering lines from $\mathcal{L}_M$.

Now consider the Case 2L. Then by (42), (34) and (35) we have that

$$L_2 \cap \mathrm{L}_\theta \in \Delta\left(P_M, \frac{2^{-14}}{1/2cn(\log^* n)^2}\right) \overset{(26)}{\subset} \Delta\left(P_M, \frac{1}{32cf(q)}\right).$$

Note that for any line $L(A,B,C) \in C_L(n,l,k)$ which go through $P_M(p,r,q)$ the distance

$$\left|\frac{A\theta + C}{B} - \frac{r}{q}\right| = \frac{|A|}{|B|}\left|\theta - \frac{p}{q}\right|$$

can differ by factor at most 16 from the same distance for line L_2. Therefore for all $L \in \mathcal{L}_M$,

$$L \cap \mathrm{L}_\theta \in \Delta(P_M, 1/2cf^{-1}(q)).$$

Whence

$$\bigcup_{L(A,B,C)\in\mathcal{L}_M} \Delta(L, cf^{-1}(|A|^*|B|^*)) \subset \Delta(P_M, cf^{-1}(q)).$$

However since $\mathcal{L}_M \subset C_L(n,l,k)$ we get by the construction of $C_L(n,l,k)$ that $\mathcal{L}_M$ has to be empty — a contradiction. Therefore the Case 2L is impossible.

Now consider the Case 3L. Let's order all the lines in $\mathcal{L}_M = (L_i(A_i,B_i,C_i))_{1\leqslant i\leqslant m_L(J)+1}$ in such a way that the sequence of the second coordinates of $L_i \cap \mathrm{L}_\theta$ is increasing. Then we have

$$|L_1 \cap \mathrm{L}_\theta - L_{m_L(M)+1} \cap \mathrm{L}_\theta| \leqslant \left|L_1 \cap \mathrm{L}_\theta - \frac{r}{q}\right| + \left|L_{m_L(J)+1} \cap \mathrm{L}_\theta - \frac{r}{q}\right|$$

$$= \left(\frac{|A_1|}{|B_1|} + \frac{|A_{m_L(M)+1}|}{|B_{m_L(M)+1}|}\right) \cdot \frac{|q\theta - p|}{q} \overset{(34),(35)}{<} c(\theta)2^{l-3k+2}R^{n-1}F(n-1) \cdot \frac{|q\theta - p|}{q}.$$

On the other hand by (14) and (35) the smallest difference between two consecutive $L_i \cap \mathrm{L}_\theta$ and $L_{i+1} \cap \mathrm{L}_\theta$ is at least

$$\frac{|q\theta - p|}{|B_i B_{i+1}|} > 2^{-2k-2}|q\theta - p|$$

and therefore

$$|L_1 \cap \mathrm{L}_\theta - L_{m_L(M)+1} \cap \mathrm{L}_\theta| > m_L(M)2^{-2k-2}|q\theta - p|.$$

By combining the upper and lower bounds for $|L_1 \cap \mathrm{L}_\theta - L_{m_L(M)+1} \cap \mathrm{L}_\theta|$ and (43) we finally get an estimate

$$m_L(M) \leqslant cn(\log^* n)^2. \tag{44}$$

3.5 Final step of the proof

Let $n \geqslant 3$. Fix an interval $J_{n-3} \in \mathcal{J}_{n-3}$. We will firstly estimate the quantity

$$\#\{P(p,r,q) \in C_P(n,l,k) \;:\; \Delta(P, cf^{-1}(q)) \cap J_{n-3} \neq \emptyset\}.$$

Split J_{n-3} into

$$K := c_1/c_2 \cdot 2^l R^2 (n-1)(n-2)[\log^*(n-1)][\log^*(n-2)]$$

subintervals $M_1, \ldots, M_K$ of equal length $c_2 2^{-l} R^{-n+1} F^{-1}(n-1)$ such that the bottom endpoint of M_i coincides with the top endpoint of M_{i+1} $(1 \leqslant i \leqslant K-1)$.

We start by constructing blocks from intervals $M_1, \ldots, M_K$. Define $B_1 := \mathbf{B}_P(M_{n_1})$, $B_2 := \mathbf{B}_P(M_{n_2}), \ldots,$ $B_t := \mathbf{B}_P(M_{n_t})$ in such a way that $n_1 := 1$ and the bottom endpoint of $\mathbf{B}_P(M_{n_i})$ coincides with the top endpoint of $\mathbf{B}_P(M_{n_{i+1}})$. By Lemma 2 for any $1 \leqslant i < t$ we have

$$\#\{P(p,r,q) \in C_P(n,l,k) \;:\; \Delta(P, cf^{-1}(q)) \cap B_i \neq \emptyset\} \leqslant m_P(M_{n_i}) + 1 \leqslant 2m_P(M_{n_i}). \tag{45}$$

Now let's consider the last block B_t. The problem is that this block is not necessarily included in J_{n-3} so we need to treat it independently. As it was discussed in Section 3.4, we have two possible cases. In Case 1P we have that for any $i \geqslant n_t$

$$\#\{P(p,r,q) \in C_P(n,l,k) \ : \ \Delta(P, cf^{-1}(q)) \cap M_i \neq \emptyset\} \leqslant 2^2.$$

By combining it with (45) we get that

$$\#\{P(p,r,q) \in C_P(n,l,k) \ : \ \Delta(P, cf^{-1}(q)) \cap J_{n-3} \neq \emptyset\} \leqslant 4K \tag{46}$$

In Case 3P we have

$$\#\{P(p,r,q) \in C_P(n,l,k) \ : \ \Delta(P, cf^{-1}(q)) \cap B_t \neq \emptyset\} \overset{(41)}{\leqslant} cn(\log^* n)^2 + 1 \overset{(17),(36)}{<} K.$$

By combining this estimate with (45) we get that

$$\#\{P(p,r,q) \in C_P(n,l,k) \ : \ \Delta(P, cf^{-1}(q)) \cap J_{n-3} \neq \emptyset\} \leqslant 3K < 4K.$$

Now estimate the number of intervals $I_{n+1} \in \mathcal{I}_{n+1}$ which are removed by $\Delta(P, cf^{-1}(q))$ where P is some interval from $C_P(n,l,k)$.

$$\begin{aligned}
\#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset\} &\leqslant \frac{|\Delta(P, cf^{-1}(q))|}{|I_{n+1}|} + 2 \\
&= \frac{2cR^{n+1}F(n+1)}{c_1 f(q) H(q)} + 2 \\
&\leqslant \frac{2cR^2 n(n+1)\,[\log^* n]\,[\log^*(n+1)]}{c_1 c(\theta) f(q) 2^l} + 2 \\
&\overset{(26)}{<} \frac{8cR^2(n+1)}{c_1 c(\theta) 2^l} + 2\,.
\end{aligned} \tag{47}$$

The upshot of the cardinality estimates (46) and (47) is that

$$\begin{aligned}
&\#\{I_{n+1} \in \mathcal{I}_{n+1} : \ J_{n-3} \cap I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \ \text{ for some } P \in C_P(n,l)\} \\
\overset{(32)}{\leqslant}\; & c_3 n \log^* n \cdot \#\{I_{n+1} \in \mathcal{I}_{n+1} \ : \ J_{n-3} \cap I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \text{ for some } P \in C_P(n,l,k)\} \\
&\leqslant (c_3 n \log^* n \cdot 4K) \left(2 + \frac{8cR^2(n+1)}{c_1 c(\theta) 2^l}\right) \\
&\leqslant 8R^2 c_3 \frac{c_1}{c_2} \cdot 2^l n^3 (\log^* n)^3 + 2^5 \frac{c_3 c}{c_2 c(\theta)} R^4 \cdot n^4 (\log^* n)^3.
\end{aligned}$$

By analogy we get the same estimate for

$$\#\{I_{n+1} \in \mathcal{I}_{n+1} \mid J_{n-3} \cap I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset \ \text{ for some } L \in C_L(n,l)\}.$$

By taking lines and points together and summing over l satisfying (30) we find that

$$\#\left\{ I_{n+1} \in \mathcal{I}_{n+1} \ \middle| \ \begin{array}{l} J_{n-3} \cap I_{n+1} \cap \Delta(P, cf^{-1}(q)) \neq \emptyset \ \text{ for some } P \in C_P(n) \text{ or} \\ J_{n-3} \cap I_{n+1} \cap \Delta(L, cf^{-1}(|A|^*|B|^*)) \neq \emptyset \ \text{ for some } L \in C_L(n) \end{array} \right\}$$

$$\leqslant 16R^2c_3\frac{c_1}{c_2}n^3(\log^* n)^3\sum_{2^l<Rn\log^* n}2^l+2^6\frac{c_3^2c}{c_2c(\theta)}R^4n^4(\log^* n)^4.$$

If $(2^{10}c(\theta))^{-1}>R^2c_1$ then in view of (36) we have that $c_2=R^2c_1$. Otherwise we have that $c_2\geqslant(2^{11}c(\theta))^{-1}$ and

$$\frac{c}{c_2c(\theta)}\leqslant 2^{11}c;\quad \frac{c_1}{c_2}\leqslant 2^{11}c_1.$$

In any case the last expression is bounded by

$$\leqslant c_4n^4(\log^* n)^4$$

where

$$c_4:=2^6\max\left\{\frac{c}{R^2c_1c(\theta)},2^{11}c\right\}\frac{(\log R+2)^2R^4}{(\log 2)^2}+16\max\{R^{-2},2^{11}c_1\}\frac{R^3(\log R+2)}{\log 2}$$

(recall that $c_3=(\log R+2)/\log 2$). In view of (18) the right hand side of this inequality is bounded by

$$25R\log R\cdot n^4(\log^* n)^4=r_{n-3,n}.$$

The upshot is that for any interval $J_{n-3}\in\mathcal{J}_{n-3}$ the number of 'bad' intervals $I_{n+1}\in\mathcal{I}_{n+1}$ which are to be removed is bounded by $r_{n-3,n}$. Therefore the desired set $\mathbf{K}_c$ is indeed a $(I,\mathbf{R},\mathbf{r})$ Cantor type set. The proof is complete.

4 Final remark.

In the proof of Theorem 2 we showed that $\mathbf{Mad}_P(f,c)\cap\mathbf{Mad}_L(f,c)\cap \mathrm{L}_\theta$ contains $(I,\mathbf{R},\mathbf{r})$ Cantor type set. It allows us to use Theorem 5 from [2]:

Theorem (BV5) *For each integer $1\leqslant i\leqslant k$, suppose we are given a Cantor set $\mathbf{K}(I,\mathbf{R},\mathbf{r}_i)$. Then*

$$\bigcap_{i=1}^{k}\mathbf{K}(I,\mathbf{R},\mathbf{r}_i)$$

is a $(I,\mathbf{R},\mathbf{r})$ Cantor set where

$$\mathbf{r}:=(r_{m,n})\quad \textit{with}\quad r_{m,n}:=\sum_{i=1}^{k}r_{m,n}^{(i)}\,.$$

Regarding the sets of the form $\mathbf{Mad}_P(f)\cap\mathbf{Mad}_L(f)\cap\mathrm{L}_\theta$, Theorem BV5 enables us to show that for any finite family $\theta_1,\dots,\theta_n$ of badly approximable numbers one can find $\alpha\in\mathbb{R}$ such the following inclusion holds simultaneously for all $1\leqslant i\leqslant n$:

$$(\alpha,\theta_i)\in\mathbf{Mad}_P(f)\cap\mathbf{Mad}_L(f).$$

Moreover the set of such numbers α is of full Hausdorff dimension. The proof is based on intersecting the corresponding Cantor type sets $K_c(i)$ associated with each set $\mathbf{Mad}_P(f,c)\cap\mathbf{Mad}_L(f,c)\cap\mathrm{L}_{\theta_i}$ for c sufficiently small and then on applying Theorem BV4 to the intersection. We will leave the details to the enthusiastic reader.

We also believe that the same fact will be true for countable collection $\{\theta_i\}$ of badly approximable numbers. However it can not be proven with existing technique.

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